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NOTE

On Maximal Closed Subsets in Association Schemes

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maximal. The result generalizes naturally the well-known (group-theoretical) fact that the one-point-stabilizer of a flag transitive automorphism group of a 2-design with $\lambda = 1$ must be a maximal subgroup. In finite group theory, there exist a lot of (important) sufficient conditions for a subgroup to be maximal. In the theory of association schemes, the condition which we offer here seems to be the first one which guarantees that a closed subset of the set of relations of an association scheme is maximal.

The starting point for the classification [2] of flag transitive automorphism groups of finite 2-designs with $\lambda = 1$ is an elementary observation due to Higman and McLaughlin. It is the well-known fact that a group which acts flag transitively on a 2-design with $\lambda = 1$ must act primitively on the set of points of the design in question; see [5, Proposition 3]. Various variants on this result were found later, either from a combinatorial point of view (cf., e.g., [3, 2.3.7(a)]) or from a group-theoretical point of view (cf., e.g., [7]).

In [8, (1.12)], it was shown that the class of groups can be viewed naturally as a distinguished class of association schemes.[1] Moreover, when going over from group theory to the theory of schemes, many basic concepts (of group theory) are generalized naturally to well-known algebraic or geometric concepts. (For example, group algebras are generalized to Hecke algebras, and Coxeter groups to Tits buildings; see [8, 9].) The purpose of the present note is to show that also the above-mentioned result of Higman and McLaughlin has a natural generalization in scheme theory.

[1] In the following, we shall say "scheme" instead of "association scheme."

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Our interest in generalizing the above-mentioned result of Higman and McLaughlin arose from the increasing difficulties we were faced with when we tried to generalize the concept of a "group action" from group theory into scheme theory. In fact, in general, it seems to be impossible to say what it means for a scheme to act on a set. In particular, the hypothesis of the above-mentioned result of Higman and McLaughlin cannot be translated literally into the language of schemes.

Against this background, it is of particular interest that the specific situation where a group acts flag transitively on a 2-design with $\lambda = 1$ can be expressed in purely group-theoretical terms. Indeed, in [5, Proposition 1], it is shown that a group G, say, which acts flag transitively on a 2-design with $\lambda = 1$ is characterized by having two subgroups H and K, say, which satisfy $H \neq G \neq K$, $KHK = G$, and $H \cup K = HK \cap KH$. Using the "complex product" which we defined in [8][2] this characterization tells us quite clearly to which objects flag transitive automorphism groups of finite 2-designs with $\lambda = 1$ must be generalized in scheme theory.

There is no problem with the translation of the conclusion of the above-mentioned result of Higman and McLaughlin into the language of scheme theory. Primitive permutation groups generalize naturally to maximal closed subsets in scheme theory. Thus, we are ready to state the desired generalization of the result of Higman and McLaughlin.

THEOREM. *Let (X, G) be a scheme, and let $K \in \mathscr{C}(G) \setminus \{G\}$ be given. Assume that there exists $H \in \mathscr{C}(G) \setminus \{G\}$ which satisfies $KHK = G$ and $H \cup K = HK \cap KH$. Then K is maximal.*

In finite group theory, there exist a lot of (important) sufficient conditions for a subgroup to be maximal. In finite scheme theory, the condition which we offer here seems to be the first one which guarantees that a closed subset of the set of relations of a scheme is maximal.

Let us now look at the notation and at the definitions used in the theorem.

Let X be a finite set.

We define

$$1 := \{(x, x) \mid x \in X\}.$$

Let $r \subseteq X \times X$ be given. We set

$$r^* := \{(y, z) \mid (z, y) \in r\},$$

[2] We shall repeat the definition further down.

and, for each $x \in X$, we define

$$xr := \{ y \in X \mid (x, y) \in r \}.$$

Let G be a partition of $X \times X$ such that $\emptyset \notin G$ and $1 \in G$.

Assume that, for each $g \in G$, $g^* \in G$. Then the pair (X, G) will be called a *scheme* if, for all $d, e, f \in G$, there exists $a_{\text{def}} \in \mathbb{N}$ such that, for all $y, z \in X$, $(y, z) \in f$ implies that $|yd \cap ze^*| = a_{\text{def}}$.

We emphasize that our definition of schemes is slightly more general than the usual definition (cf., e.g., [1]). Our notion of schemes coincides with Higman's notion of homogeneous coherent configurations; see [4].

For the remainder of this note, we shall assume that (X, G) is a scheme.

For all $E, F \subseteq G$, we define

$$EF := \left\{ g \in G \mid \sum_{e \in E} \sum_{f \in F} a_{efg} \neq 0 \right\}.[3]$$

A subset F of G is called *closed* if $FF \subseteq F \neq \emptyset$.

We shall denote by $\mathscr{C}(G)$ the set of all closed subsets of G.[4]

Let $H \in \mathscr{C}(G)$ be such that $H \neq G$. Then H is called *maximal* if, for each $K \in \mathscr{C}(G)$, $H \subseteq K$ implies that $K \in \{H, G\}$.

So far, we have given the notation and the definitions used in the theorem. Let us now collect the lemmata which will lead to the proof of the theorem.

Our first result is a generalization of a well-known fact in group theory. It is due to R. Dedekind. Its (simple) proof is given in [8, (1.3)(iii)].

LEMMA 1. *Let $H \in \mathscr{C}(G)$, let $E \subseteq H$, and let $F \subseteq G$ be given. Then we have $H \cap EF = E(H \cap F)$ and $H \cap FE = (H \cap F)E$.*

Let $H \in \mathscr{C}(G)$ be given. For each $x \in X$, we define

$$xH := \bigcup_{h \in H} xh.$$

We set

$$n_H := \sum_{h \in H} a_{hh^*1}$$

and

$$X/H := \{ xH \mid x \in X \}.$$

[3] It is straightforward that, for all $D, E, F \subseteq G$, $(DE)F = D(EF)$.

[4] Note that, for all $H, K \in \mathscr{C}(G)$, $H \cap K \in \mathscr{C}(G)$.

Also, our second lemma is a generalization of an important fact in group theory. For a proof of its first part, see [8, (1.1)]. Its second part is straightforward; the third part follows from the first two parts.

LEMMA 2. *Let $H \in \mathscr{C}(G)$ be given. Then we have the following.*

(i) *X/H is a partition of X.*

(ii) *For each $x \in X$, $|xH| = n_H$.*

(iii) *$n_G = n_H |X/H|$.*

Let $Y \subseteq X$. For each $g \in G$, we define

$$g_Y := g \cap (Y \times Y).$$

For each $F \subseteq G$, we set

$$F_Y := \{f_Y \mid f \in F\}.$$

For each $x \in X$ and, for each $H \in \mathscr{C}(G)$, we set

$$(X, G)_{xH} := (xH, H_{xH}).$$

The first part of the following lemma is [8, (1.2)(i)]. Its second part follows easily from [8, (1.2)(ii)], the third part is an immediate consequence of the second one.

LEMMA 3. *Let $x \in X$, and let $H \in \mathscr{C}(G)$ be given. Then we have the following.*

(i) *$(X, G)_{xH}$ is a scheme.*

(ii) *For all $E, F \subseteq H$, $E_{xH} F_{xH} = (EF)_{xH}$.*

(iii) *For each $F \subseteq H$, $F_{xH} \in \mathscr{C}(H_{xH})$ if and only if $F \in \mathscr{C}(G)$.*

Let us look back to Lemma 2(i). Switching to a geometric language, this lemma says that, for each $\mathscr{H} \subseteq \mathscr{C}(G)$ with $\mathscr{H} \neq \varnothing$,

$$\mathscr{C}(X, \mathscr{H}) := (X, \{X/H \mid H \in \mathscr{H}\})$$

is a chamber system in the sense of J. Tits [6].

In the following lemma, we shall consider 2-designs as chamber systems. The idea of viewing particular geometries as chamber systems is discussed in Section 2.2 of [6]. For our purposes, it is enough to know the following. The "points" of the design in question will be the elements of X/K, the "blocks" will be the elements of X/H. "Incidence" is defined as "having non-empty intersection."

LEMMA 4. *Let H, $K \in \mathscr{C}(G) \backslash \{G\}$ be given. Assume that $KHK = G$ and that $H \cup K = HK \cap KH$. Then we have the following.*

(i) *$\mathscr{C}(X, \{H, K\})$ is a 2-design with $\lambda = 1$.*

(ii) *We have*

$$\frac{n_G}{n_K} = 1 + \left(\frac{n_H}{n_{H \cap K}} - 1\right) \frac{n_K}{n_{H \cap K}}.$$

(iii) $n_H \leqslant n_K$.

(iv) *Let $M \in \mathscr{C}(G)$ be such that $K \subseteq M$. Then $H \cap M \neq M$, $K(H \cap M) K = M$, and $(H \cap M) \cup K = (H \cap M) K \cap K(H \cap M)$.*

Proof. (i) First of all, we obtain from Lemma 2(i), (ii) that, for each $x \in X$,

$$|\{yH \mid xK \cap yH \neq \emptyset\}| = \frac{n_K}{n_{H \cap K}}$$

and

$$|\{yK \mid xH \cap yK \neq \emptyset\}| = \frac{n_H}{n_{H \cap K}}.$$

Let y, $z \in X$ be such that $yK \neq zK$. Let $g \in G$ be such that $(y, z) \in g$.

Since we are assuming that $KHK = G$, we find v, $w \in X$ such that $v \in yK$, $w \in vH$, and $z \in wK$. It follows that $yK \cap vH \neq \emptyset$ and $zK \cap vH \neq \emptyset$.

Assume now that there exists $x \in X$ such that $yK \cap xH \neq \emptyset$ and $zK \cap xH \neq \emptyset$. (We shall be done if we succeed in showing that $vH = xH$.)

By Lemma 2(i), we may assume without loss of generality that $y \in vH$ and that $z \in xH$. It follows that $g \in HK \cap KH$. Thus, by hypothesis, $g \in H \cup K$.

If $g \in K$, Lemma 2(i) yields $yK = zK$, contrary to the choice of y, $z \in X$.

Therefore, we must have that $g \in H$. It follows that $yH = zH$. But, as $y \in vH$, Lemma 2(i) says that $yH = vH$. Similarly, $z \in xH$ implies that $zH = xH$. Therefore, we obtain that $vH = xH$.

(ii) follows from (i), Lemma 2(i), and [3, (2.1.5)].

(iii) From (i) and "Fisher's inequality" [3, 1.3.8] we know that $|X/K| \leqslant |X/H|$. Thus, the claim follows from Lemma 2(iii).[5]

[5] Let us mention here that the inequality $n_H \leqslant n_K$ can be derived directly from the hypotheses that $H \neq G \neq K$, that $KHK = G$, and that $H \cup K = HK \cap KH$. This means that our theorem can be proved completely within scheme theory.

(iv) Suppose, by way of contradiction, that $H \cap M = M$. Then $M \subseteq H$. Thus, our hypothesis that $K \subseteq M$ leads to $K \subseteq H$. It follows that $H = KHK$. On the other hand, we are assuming that $KHK = G$, so that we obtain $H = G$. Since this contradicts the hypothesis of the lemma, we must have that $H \cap M \neq M$.

Using Lemma 1, we conclude that

$$K(H \cap M)K = (KH \cap M)K = KHK \cap M = M.$$

(Recall that we are assuming that $KHK = G$.)

Using Lemma 1 once again, we also obtain that

$$(H \cap M) \cup K = (H \cup K) \cap M = HK \cap M \cap KH = (H \cap M)K \cap K(H \cap M).$$

(Recall that we are assuming that $H \cup K = HK \cap KH$.) This finishes the proof of Lemma 4.

The reader might be interested to know that the converse of Lemma 4(i) can be derived easily from the proof of [8, (2.4)].

Let us now finish this note by proving the theorem.

Let $M \in \mathscr{C}(G)$ be such that $K \subseteq M$ and $K \neq M$. We shall be done if we succeed in showing that $M = G$.

From Lemma 4(iv) and Lemma 3(ii), (iii) we conclude that, for each $x \in X$, the scheme $(X, G)_{xM}$ together with $(H \cap M)_{xM}$ and K_{xM} satisfies the hypotheses of Lemma 4.

Let us abbreviate

$$\eta := \frac{n_H}{n_{H \cap K}},$$

$$\kappa := \frac{n_K}{n_{H \cap K}},$$

and

$$\sigma := \frac{n_{H \cap M}}{n_{H \cap K}}.$$

Then Lemma 4(ii) yields

$$\frac{n_M}{n_K} = 1 + (\sigma - 1)\kappa$$

and

$$\frac{n_G}{n_K} = 1 + (\eta - 1)\kappa.$$

(Apply Lemma 4(ii) first to $(X, G)_{xM}$ and then to (X, G).)

Now recall that, by Lemma 2(iii), n_M divides n_G. Thus, $1 + (\sigma - 1)\kappa$ divides $1 + (\eta - 1)\kappa$. Let $n \in \mathbb{N} \setminus \{0\}$ be such that

$$n[1 + (\sigma - 1)\kappa] = 1 + (\eta - 1)\kappa.$$

Then

$$n - 1 = \kappa[\eta - 1 - n(\sigma - 1)].$$

Suppose, by way of contradiction, that $\sigma = 1$. Then, by definition, $n_{H \cap K} = n_{H \cap M}$. Since $K \subseteq M$, this yields $H \cap K = H \cap M$. This means that $H \cap M \subseteq K$. It follows that $K = K(H \cap M)K$. On the other hand, we know from Lemma 4(iv) that $K(H \cap M)K = M$, so that we obtain $K = M$. Since this contradicts the choice of $M \in \mathscr{C}(G)$, we must have that $2 \leqslant \sigma$.

We now shall prove that $n = 1$. Suppose that $2 \leqslant n$. Then, as $2 \leqslant \sigma$, the last of the above equations yields $n \leqslant \eta - 1$ and $\kappa \leqslant n - 1$. It follows that $\kappa \leqslant \eta - 1$, which means that $n_K \leqslant n_H - 1$. This contradicts Lemma 4(iii).

Thus, we have shown that $n = 1$. This time, the last of the above equations yields that $\sigma = \eta$. It follows that $n_{H \cap M} = n_H$. This means that $H \cap M = H$, in other words, that $H \subseteq M$. Since we are assuming that $KHK = G$ and that $K \subseteq M$, this implies that $M = G$.

This finishes the proof of the theorem.

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